THE

MATHEMATICAL GAZETTE.

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LONDON:
GEORGE BELL & SONS, PORTUGAL STREET, LINCOLN'S INN,
AND BOMBAY.

VOL. III. JULY, 1905. No. 52.

ON THE DEVELOPMENT OF MATHEMATICAL ANALYSIS AND ITS RELATIONS TO SOME OTHER SCIENCES.*

ONE of the objects of a Congress such as this is to exhibit the connecting links of the different departments of Science, taking that term in its widest acceptation. Hence the organisers of this meeting have taken care that the relations between the different sections shall be exhibited in broad relief. In undertaking a sketch of this kind, which must necessarily be somewhat indefinite in character, we must forget that all is in all; as far as algebra and analysis are concerned, the follower of Pythagoras would be disconcerted by the extent of his task, remembering the celebrated formula of the School: "Things are numbers." From this point of view my subject would be inexhaustible. But I shall make no such pretensions, and for the best of reasons. By passing in rapid review the development of our science throughout the ages, and particularly in the last century, I hope to be able to give an adequate indication of the part played by mathematical analysis in its relations to other sciences.

I.

I might naturally be expected to begin with the concept of the whole number, but this concept is not solely logical in its nature; it is also historical and psychological, and would involve us in too many discussions. Unfathomable depths have been revealed in the course of the study of the concept of number. For instance, no solution has yet been found of the problem under which of the two forms, the cardinal and the ordinal, does the idea of number first present itself, which of the two is anterior to the other, *i.e.* is the idea of number

* An address delivered at the Congress of Science and Art, St. Louis, U.S.A., September 22nd 1904, and translated for the *Gazette* by kind permission of the author.
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properly so called anterior to that of order, or *vice versâ*. In these questions geometers who are logicians seem to pay too little attention to psychology and the lessons to be derived from uncivilised races. Perhaps there is no general answer to the question, for the answer varies according to races and mental powers. In this connection I have sometimes thought of the distinction between auditives and visuals, the auditives being connected with the ordinal theory, and the visuals with the cardinal. But I must not linger on ground so full of pitfalls. I am afraid that our modern school of logicians has a difficulty in finding common ground with ethnographers and biologists; the latter, in questions of origin, are always dominated by the idea of evolution, and, to more than one among them, logic is but the summary of ancestral experience. Mathematicians have even been reproached for laying down in principle that there is a human mind in some measure external to things, and that it has a logic of its own. Whatever may be thought of this, the idea has been very useful, if not indispensable, to the progress of mathematical science; and certainly whether it evolved in the course of prehistoric ages or not, this logic of the human mind was firmly fixed in the oldest geometrical schools in Greece; the works of those schools appear to have been the first code of that logic, a fact indicated by the story of Plato writing on the door of his school: "Let none enter here if he be not a geometer."

A long time before the quaint word *Algebra* was derived from the Arabic, a word which expresses the operation by which equalities are reduced to a certain canonical form, the Greeks were unconsciously in the possession of algebra. We cannot even imagine relations closer than those which connect their algebra and their geometry, or rather, if need arose, it would be difficult to classify their geometrical algebra, in which they reasoned not on numbers, but on magnitudes. Among the Greeks, too, we find a geometrical arithmetic; and one of the most interesting phases of its development is the discussion which took place among the followers of Pythagoras on the difference between number and magnitude, *apropos* of irrationals. Although the Greeks cultivated the abstract study of numbers, which they called arithmetic, their speculative spirit found but little attraction in practical calculations, which they called logistic. In remote antiquity the Egyptians, Chaldeans, and later the Hindus and the Arabs, carried the science of computation to considerable lengths. To this they were forced by practical requirements. Logistic preceded arithmetic just as land surveying and geodesy opened the way for geometry. In a like manner trigonometry was developed under the influence of the increasing requirements of astronomy. The initial stages of the history of science show a close relation

between pure and applied mathematics, and in the course of this address we shall constantly find this to be the case.

So far our domain has been that which is called in current language elementary arithmetic and algebra. But as soon as the incommensurability of certain magnitudes was recognised, the idea of infinity made its appearance, and from the days of Zeno's paradoxes on the impossibility of motion, geometrical progressions had to be dealt with. The processes of exhaustion which are found in Eudoxus and in Euclid really belong to the integral calculus, and Archimedes evaluates definite integrals. Mechanics also makes its appearance in his treatise on the quadrature of the parabola, for he first of all finds the area of the segment limited by an arc of the parabola and its chord, by means of the theorem of moments. This is the first example of those relations between mechanics and analysis which have since been continually developed. The infinitesimal method of the Greek geometers in treating volumes raised questions the interest of which has not even yet been exhausted. In plane geometry two equivalent polygons are equal by addition or subtraction, *i.e.* they may be decomposed into congruent triangles, or they may be regarded as the differences of polygons which may be so decomposed. It is not the same in the case of the geometry of space, and it has only recently been known that stereometry cannot, like planimetry, be independent of the processes of exhaustion or limit, which require the axiom of continuity or Archimedes' axiom. Without dwelling on this further, this glance into antiquity shows us how algebra, arithmetic, geometry, and elementary forms of the integral calculus and mechanics were intermingled to such an extent that it is impossible to separate their early history.

In the middle ages and in the renaissance the geometrical algebra of the ancients is separated from geometry. Little by little algebra, properly so called, became independent, with its symbolism and its notation more and more perfected. Thus is created this wonderfully clear language which economises thought and makes further progress possible. It is now also that distinct divisions become organised. Trigonometry, which in antiquity was but the handmaid of astronomy, begins to develop independently, and about the same time the logarithm appears, and essential elements are thus brought into evidence.

II.

In the 17th century the analytical geometry of Descartes—distinguished from what I just now called the geometrical algebra of the Greeks by the general and systematic ideas upon which it is founded—and the newly-born science of dynamics, proved to be the origin of very great progress in analysis. When Galileo, starting from the hypothesis that the velocity

of heavy falling bodies is proportional to the time, deduces from it the law of the distance traversed, verifying it later by experiments, he is following the path already trodden by Archimedes, a path which was to be followed in time to come by Cavalieri, Fermat, and others, down to the time of Newton and Leibnitz. The integral calculus of the Greek geometers is found again in the kinematics of the great Florentian physicist. As for the calculus of derivatives or differentials, it was laid down with precision in connection with the drawing of tangents. In reality, the origin of the idea of derivatives lies in the vague sense of the mobility of things and of the more or less extreme rapidity with which phenomena take place. This is well expressed in the terms *fluentes* and *fluxions* used by Newton, which we might suppose are borrowed from Heraclitus of old.

The standpoints of the founders of the science of motion, Galileo, Huygens, and Newton, had an enormous influence on the direction of mathematical analysis. In Galileo it was an intuition of genius to note that in natural phenomena the circumstances which determine motion produce acceleration. It was bound to lead him to lay down the principle that the rapidity with which the dynamical state of a system changes depends in a definite manner on its statical state alone. In a more general manner it was postulated that the infinitesimal changes, of whatever nature they may be, which take place in a system of bodies depend solely on the present state of that system. How far are the exceptions apparent or real? This is a question which was only raised at a later stage, and one which I pass over for the moment. From the principles enunciated is disengaged a point of great importance to the analyst. The phenomena are expressed by differential equations—equations which may be formed when observation and experiment have detected in each category of phenomena certain physical laws. One understands the unbounded hopes to which such results must give rise. As Bertrand says in the preface to his *Traité*, the initial successes were at first so great that it might be supposed that all the difficulties of science were already surmounted; and it was thought that geometers, no longer being distracted by the elaboration of pure mathematics, would be able to turn their thoughts exclusively to the study of natural laws. This was a gratuitous assumption that the problems of analysis presenting themselves would offer no very serious difficulties. But in spite of the disillusions which the future was to bring, the important point remained that these problems had assumed a precise form, and that the difficulties which were to be surmounted could be classified. Here, then, was an immense

advance, one of the greatest that the human mind has ever made. In this manner we also learn why the theory of differential equations was to acquire considerable importance.

I have anticipated somewhat, in presenting things in such an analytical form. In all this progress geometry played its part. Huygens, for instance, always followed the ancients by preference, and his *Horologium Oscillatorium* is based on both infinitesimal geometry and mechanics. In the same way the methods followed in Newton's *Principia* are synthetic. But it is with Leibnitz in particular that science begins to move along those paths which were to lead it to what we call mathematical analysis. It is Leibnitz who in the last years of the 17th century is the first to employ the word *function*. By his power of systematisation, by the numerous problems which he and his disciples, James and John Bernouilli, treated, he showed definitively the power of the methods to which a long series of thinkers had in turn contributed ever since the far-off days of Eudoxus and Archimedes.

The 18th century showed the extreme fertility of the new methods. It was a curious age. It was an age of mathematical duels in which geometer launched defiance at geometer, in which the conflict was not always without bitterness, when the followers of a Leibnitz and a Newton entered the lists. From the purely analytical point of view the classification and the study of simple functions is particularly interesting; the idea of function on which analysis is based is thus gradually developed. Euler's celebrated researches at this time held an important position. However, the numerous problems which presented themselves to mathematicians scarcely left them time to scrutinise principles; the foundations of the doctrine are slowly elucidated, and the epigram attributed to d'Alembert: "Go on, and faith will come to you," is very characteristic of this age. Of all the problems raised at the end of the 17th century, or during the first half of the 18th, it will suffice to recall the problems of isoperimetry, which were to give birth to the calculus of variations. I prefer to dwell on the more intimate relations still between analysis and mechanics when, after the inductive period of the early days of dynamics, we reach the deductive period in which an endeavour was made to give a definite form to principles. Mathematical and formal development then played the essential rôle, and the language of analysis was indispensable to the widest extension of these principles. There are times in the history of sciences, and perhaps of societies, in which the mind is sustained and carried forward by the words and the symbols which it has created, and when generalisations present themselves with a minimum of effort. This was in

particular the part played by analysis in the formal development of mechanics. Allow me here to interpose a remark. It is often said that there is nothing in an equation except what we put into it. It is easy to reply, in the first place, that the new form under which one finds things is often itself an important discovery. But we may go further. Analysis by the simple play of its symbols may suggest generalisations going far beyond the original limits. Thus it was with the principle of virtual velocities, the first idea of which sprang from the simplest mechanisms. The analytical form which translated it suggested extensions which led far from the point of departure. In one sense even, it is not fair to say that analysis has created nothing, for these more general conceptions are the work of analysis. We have another example in Lagrange's system of equations. Here the transformations of the calculus have given the type of differential equations to which we now tend to reduce the notion of mechanical explanation. In science there are few examples comparable with this of the importance of the form of an analytical relation, and of the power of generalisation of which it may be capable. It is very clear that in each case the generalisation suggested must be stated precisely by an appeal to observation and experiment. Finally, it is the calculus again which will seek out remote consequences to submit them to the same tests—but that is an order of ideas which I may not dwell upon here.

Under the impulse of the problems suggested by geometry, mechanics, and physics, we see developing, or springing into being, nearly all the great divisions of analysis. First we meet with equations with a single independent variable. Presently partial differential equations appear in the treatment of vibrating strings, of the mechanics of fluids, and of the infinitesimal geometry of surfaces. Here was quite a new analytical world. The very origin of the problems treated was of assistance and did not permit of much wandering from the right path in the early stages; and in the hands of Monge geometry rendered useful aid to these newly-born theories. But of all the applications of analysis none at that time was more brilliant than the problems of celestial mechanics, which were suggested by the knowledge of the laws of gravitation, and with which are connected the names of our most famous geometers. Theory never had a finer triumph; perhaps, even, it might be added that this triumph was too complete, for it was at this time especially that were conceived for the future of natural philosophy the premature hopes to which I have alluded During all this period, especially in the second half of the 18th century, what strikes us with admiration, and also is the cause of some little confusion, is the extreme importance of the applications realised, at a time when

the foundations of pure theory appeared so uncertain. We see it when certain questions are raised, such as the degree of arbitrariness in the integral of vibrating strings, which gives rise to an interminable and somewhat inconclusive discussion. Lagrange realised these insufficiencies when he published his theory of analytical functions in which he endeavours to give precise foundations to analysis. It is impossible to admire too highly the wonderful presentiment he had of the part that the functions which we now, with him, call analytical functions were to play in the future. But we stand astonished, it must be confessed, when we read the proof that he thought he had given of the possibility of the development of a function in Taylor's series. Requirements in questions of pure analysis were but few and far between in those days. Trusting in intuition, men were content with certain probabilities, and agreed implicitly on certain hypotheses which it seemed unnecessary to formulate explicitly. In reality, they trusted in the solidity of ideas which had shown themselves fertile on so many occasions, and this is practically what d'Alembert expressed in his epigram. The necessity of rigour in mathematics has had its successive approximations, and in this respect our science does not possess the absolute character attributed to it by so many.

III.

We have now reached the early years of the 19th century. As I explained before, the great majority of analytical researches in the 18th century were occasioned by problems in geometry, and especially in mechanics or in physics, and so far we have found but little trace of the logical and aesthetic tendencies which are to give such a different appearance to so many mathematical treatises, especially in the last two-thirds of the 19th century. But I must not anticipate. After numerous examples of the influence of physics on the development of analysis, we shall meet with yet another, and that one of the most memorable, in Fourier's theory of heat.

Fourier begins by forming the partial differential equations which govern temperature. What are for such equations the conditions at the limits which enable us to obtain a solution? To Fourier's mind the conditions were suggested by the physical problem, and the methods he followed have served as models to the physicist-geometers of the first half of the last century. One of them consists in forming series with certain simple solutions. Fourier thus obtained his first types of development more general than the trigonometrical developments, as in the problem of the cooling of a sphere in which he applies his theory to the terrestrial globe, and seeks the law of variations of temperature in the soil, trying to obtain numerical applications.

In the presence of these grand results we can understand the enthusiasm of Fourier which glows in every line of his introductory remarks. Speaking of mathematical analysis, he says: "There can be no more universal or more simple language, no language more exempt from error and obscurity, *i.e.* more worthy to express the invariable relations of natural objects. Considered from this point of view it is as far extended as nature itself; it defines all sensible relations, it measures time, space, force and temperature. This difficult science is formed slowly, but it never loses a principle that it has once acquired. It ever grows and becomes strong, though surrounded by errors of the human mind." This praise is magnificent, but here we see signs of the tendency which makes of analysis but an auxiliary, however incomparable, to the natural sciences, a tendency of thought in conformity, as we have seen, with the development of science during the two preceding generations; but here we reach an epoch at which new tendencies make their appearance. Poisson, in a report on the *Fundamenta*, recalled the reproach levelled by Fourier at Abel and Jacobi, that they had not by preference devoted their attention to the motion of heat. Jacobi wrote to Legendre: "It is true that M. Fourier believed that the main aim of mathematics was public utility and the explanation of natural phenomena; but a philosopher of his ability ought to have known that the sole aim of science is the honour of the human intellect, and that on this ground a problem in numbers is as important as a problem on the system of the world." No doubt this was also the opinion of the great geometer of Göttingen, who called mathematics the queen of the sciences and arithmetic the queen of mathematics. It would be ridiculous to consider these two tendencies as directly opposed; the harmony of our science is in their synthesis. The moment was to arrive at which the necessity of inspecting the foundations of the edifice became evident, at which an inventory had to be made of the riches accumulated, adopting at the same time a more critical attitude. Mathematical thought was to acquire fresh impetus by a rigorous examination of mathematical principles. The problems are for a time exhausted, and it is not well that all investigation should proceed along one and the same line. Besides, differences and paradoxes which remained unexplained required further progress in pure theory. The path along which pure theory was to move was traced in its broad lines, and it could continue its journey independently, without necessarily losing contact with the problems suggested by geometry, mechanics, and physics. At the same time more interest was to be attached to the philosophical and artistic side of mathematics, trusting in a kind of pre-established harmony between our logical and aesthetic satisfactions and the requirements of future applications.

Let us rapidly recall a few points in the history of the revision of principles in which Gauss, Cauchy, and also Abel were workers at this early stage. Precise definitions of continuous functions and their most immediate properties, and simple rules for the convergence of series were formulated; and there soon was established under very general conditions the possibility of trigonometrical development, thus giving legal sanction to the boldness of Fourier. Certain geometrical intuitions relating to areas and arcs gave place to rigorous proofs. The geometers of the eighteenth century had necessarily tried to take into account the degree of generality of the solution of ordinary differential equations. Their resemblance to equations of finite differences readily led to the result; but the proofs obtained in this way will not bear close examination. Lagrange in his *Leçons sur le Calcul des Fonctions* had introduced more precision, and starting from Taylor's series, he saw that the equation of order m leaves indeterminate the function and its $m-1$ first derivatives for the initial value of the variable; but we are not surprised that Lagrange has not considered the question of convergence. In 20 or 30 years, demands for rigour in proofs had grown. We know that the two preceding methods of demonstration are susceptible of all the precision that is requisite. As for the first, there was no need of any new principle; and as for the second, the theory had to be developed along new lines. So far, functions and variables had been real. The consideration of complex variables now extended the field of analysis. The functions of a complex variable with a single derivative are necessarily developable in a Taylor's series. Thus we fall back on that mode of development, the interest of which had been recognised by the author of the *Théorie des Fonctions analytiques*, but the importance of which could not be fully realised as long as we limited ourselves to real variables. They also owe the great part they have not ceased to play to the facility with which they can be handled, and their convenience in calculations. The general theorems of the theory of analytical functions afforded a precise answer to questions which until then had remained undecided, such as the degree of generality of the integrals of differential equations. It became possible to complete the proof sketched by Lagrange for an ordinary differential equation; and precise theorems were established for a partial differential equation, or a system of such equations. The results obtained, important as they are, do not completely solve the different questions that may be raised; for in mathematical physics, and even in geometry, the conditions for the limits are susceptible of forms so varied that the problem known as Cauchy's problem often appears in a very restricted form. I shall come back later to this important point. EMILE PICARD.

(*To be continued.*)

I.—EXPANSIONS OF TRIGONOMETRICAL FUNCTIONS.

In the following articles the method of Averages, used in the Oct. 1903 issue of the *Gazette* to effect the expansion of the simple Algebraical functions, is employed for the Trigonometrical Expansions and for Taylor's Theorem, and the results are applied to the question of convergence. The method was adopted as the simplest I could devise for even the Binomial, subject to the condition that no infinite series should be used unless the limits of error after a finite number of terms were explicitly stated. I would suggest that this condition might well be laid down in a school course, as the uncertainty that comes over our Algebra as soon as we leave Geometrical Progressions 'to infinity' delays a clear grasp of Mathematics more than a fuller consideration of the simpler infinite series would. The method is, of course, essentially integral calculus, but I have found—and I hope my subjects have not been specially favourable—that the proofs can be readily followed by students to whom the ideas and notation of the Differential and Integral will not be familiar for a considerable time, and in whom perhaps they will never inspire a sense of absolute security.

1. *Expansion of the sine.*

$$\sin x - \sin y = 2 \sin \frac{x-y}{2} \cos \frac{x+y}{2},$$

which $$= 2 \tan \frac{x-y}{2} \cos \frac{x-y}{2} \cos \frac{x+y}{2}.$$

Now, if the angles are in circular measure,

$$2 \sin \frac{x-y}{2} < (x-y), \quad \text{and } 2 \tan \frac{x-y}{2} > (x-y).$$

Moreover, if x and y are less than $\frac{\pi}{2}$,

$$\cos \frac{x+y}{2} < \cos y, \quad \text{and } \cos \frac{x-y}{2} \cos \frac{x+y}{2} = \tfrac{1}{2}(\cos x + \cos y) > \cos x;$$

$$\therefore \sin x - \sin y \overset{*}{<} (x-y) \begin{vmatrix} \cos y \\ \cos x \end{vmatrix}.$$

Now consider the interval between 0 and a broken up into N equal intervals, the intermediate angles being

$$\frac{a}{N}, \frac{2a}{N}, \cdots \frac{(N-2)a}{N}, \frac{(N-1)a}{N},$$

which we may write shortly

$$a_1, a_2, \cdots a_{N-2}, a_{N-1}.$$

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* "Lies between."

Then, $\sin a \equiv (\sin a - \sin a_{N-1}) + (\sin a_{N-1} - \sin a_{N-2}) + \ldots + (\sin a_1 - \sin 0)$,

that is, $$\sin a = \sum_{i=1}^{i=N} (\sin a_i - \sin a_{i-1}).$$

Therefore, provided $a < \frac{\pi}{2}$ (so as to ensure that $\cos a_{i-1}$ is always greater than $\cos a_i$),

$$\sin a < \sum_{1}^{N} \frac{a}{N} \begin{vmatrix} \cos a_{i-1} \\ \cos a_i \end{vmatrix},$$

that is, $$< \frac{a}{N} \begin{vmatrix} \cos 0 + \cos a_1 + \cos a_2 + \ldots + \cos a_{N-2} + \cos a_{N-1} \\ \cos a_1 + \cos a_2 + \ldots \qquad\qquad + \cos a_{N-1} + \cos a \end{vmatrix}. \quad (1)$$

Now the difference between the upper and lower limits is

$$\frac{a}{N}(\cos 0 - \cos a) \text{ or } \frac{a}{N} \cdot 2 \sin^2 \frac{a}{2}, \text{ which } < \frac{a^3}{2N}.$$

But N is quite independent of all the other magnitudes concerned, and therefore we may consider it indefinitely increased, and may neglect this small error, writing for either of the limits in (1) $a \cdot \underset{\theta=0}{\overset{\theta=1}{\text{Av}}} (\cos \theta a)$, θ representing the fractions $\frac{1}{N}, \frac{2}{N}, \ldots$ and 'Av' the average of the values when N is *indefinitely* increased.

Thus $$\sin a = a \cdot \underset{0}{\overset{1}{\text{Av}}} \cos \theta a. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{s})$$

An exactly similar piece of work leads to

$$\cos a - 1 = -a \cdot \underset{0}{\overset{1}{\text{Av}}} \sin \theta a. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots (\text{c})$$

Hence, substituting for $\cos \theta a$ in (s) by means of (c),

$$\sin a = a \underset{\theta=0}{\overset{\theta=1}{\text{Av}}} \left[1 - \theta a \underset{\theta'=0}{\overset{\theta'=1}{\text{Av}}} \sin \theta\theta' a \right] = a - a^2 \cdot \underset{0}{\overset{1}{\text{Av}}} \theta \cdot \underset{0}{\overset{1}{\text{Av}}} \sin \theta\theta' a.$$

Using the same process on $\sin \theta\theta' a$, and introducing subscripts to θ, we get

$$\sin a = a - a^3 \cdot \underset{0}{\overset{1}{\text{Av}}} \theta_1^2 \cdot \underset{0}{\overset{1}{\text{Av}}} \theta_2 + a^4 \cdot \underset{0}{\overset{1}{\text{Av}}} \theta_1^3 \cdot \underset{0}{\overset{1}{\text{Av}}} \theta_2^2 \cdot \underset{0}{\overset{1}{\text{Av}}} \theta_3 \cdot \underset{0}{\overset{1}{\text{Av}}} (\sin \theta_1\theta_2\theta_3\theta_4 a).$$

Now, by a simple process,* we find that $\underset{0}{\overset{1}{\text{Av}}} \theta^n = \frac{1}{n+1}$;

$\therefore \sin a = a - \frac{a^3}{3} + a^4$. (the same expression as before).

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* *Gazette*, July, 1903, § v.

By repetition of the process we obtain, if n is odd,

$$\sin a = a - \frac{a^3}{\lfloor 3} + \frac{a^5}{\lfloor 5} - \ldots \pm \frac{a^n}{\lfloor n} \mp \text{ an error which is}$$

$$a^{n+1} \,.\, \text{Av}\, \theta_1{}^n \,.\, \text{Av}\, \theta_2{}^{n-1} \ldots \text{Av}\, \theta_n \,.\, \text{Av} \sin(\theta_1\theta_2 \ldots \theta_n\theta_{n+1}a).$$

The value of this error can be obtained by the consideration that, as every θ is a fraction, the sine that occurs is less than $\sin a$.

$\therefore$ the error lies between $\dfrac{a^{n+1}}{\lfloor n+1} \sin a$ and 0.

By the use of equation (S) above, it is equally easy to show that the error lies between $\dfrac{a^{n+2}}{\lfloor n+2} \begin{vmatrix} 1 \\ \cos a \end{vmatrix}$, n still being odd.

In either case it diminishes indefinitely as n increases.

2. *The expansion of the cosine* follows from equation (C), or can be established *à priori* by similar work.

$$\cos a = 1 - \frac{a^2}{\lfloor 2} + \frac{a^4}{\lfloor 4} - \ldots \pm \frac{a^n}{\lfloor n} \; (n \text{ even}) \mp \text{ an error}$$

which lies between $\dfrac{a^{n+1}}{\lfloor n+1} \begin{vmatrix} \sin a \\ 0 \end{vmatrix}$, or between $\dfrac{a^{n+2}}{\lfloor n+2} \begin{vmatrix} 1 \\ \cos a \end{vmatrix}$.

[NOTE. For the sake of simplicity these expansions are here proved only for acute angles: it is shown in § II., that a slight modification of the proof makes it general.]

3. *Expansion of the inverse tangent.*

We have $\tan x - \tan y = \dfrac{\sin(x-y)}{\cos x \,.\, \cos y} = \tan(x-y) \,.\, \dfrac{\cos(x-y)}{\cos x \cos y}$.

Hence, $\tan x - \tan y < (x-y) \begin{vmatrix} \sec^2 x \\ \sec^2 y \end{vmatrix}$.

[We could get from this, as before,

$$\tan a = a \,.\, \overset{1}{\underset{0}{\text{Av}}}\, (\sec^2 \theta a) = a + a^2 \overset{1}{\underset{0}{\text{Av}}}\, (\tan^2 \theta a);$$

but the square under the average introduces difficulties.]

Since we may write the above in the form

$$(x-y) < (\tan x - \tan y) \begin{vmatrix} \cos^2 y \\ \cos^2 x \end{vmatrix},$$

we have $\tan^{-1} a - \tan^{-1} b < (a-b) \begin{vmatrix} (1+b^2)^{-1} \\ (1+a^2)^{-1} \end{vmatrix}$

leading, by the usual process, to $\tan^{-1} x = x \,.\, \overset{1}{\underset{0}{\text{Av}}}\, (1+\theta^2 x^2)^{-1}$.

Now $\dfrac{1 \mp \theta^{2n}x^{2n}}{1+\theta^2x^2} = 1-\theta^2x^2+\ldots\pm\theta^{2n-2}x^{2n-2}$, the sign depending on n even or odd.

$$\therefore \tan^{-1}x = x-\frac{x^3}{3}+\frac{x^5}{5}\ldots\mp\frac{x^{2n-1}}{2n-1}\pm\text{an error } x^{2n+1}\,.\,\text{Av}\,\frac{\theta^{2n}}{1+\theta^2x^2}.$$

This error is less than*

$$x^{2n+1}\,.\,\text{Av}\,\theta^{2n}\,.\,\text{Av}\,(1+\theta^2x^2)^{-1},$$

which $$= \frac{x^{2n+1}}{2n+1}\cdot\frac{\tan^{-1}x}{x},$$

and is greater than $$\frac{x^{2n+1}}{2n+1}\cdot\frac{1}{1+x^2},$$

and therefore diminishes indefinitely if x is less than 1.

4. *Expansion of the inverse sine and cosine.*

Since $$\sin x-\sin y < (x-y)\begin{vmatrix}\cos y\\ \cos x\end{vmatrix},$$

$$\sin^{-1}a-\sin^{-1}b < (a-b)\begin{vmatrix}(1-a^2)^{-\frac{1}{2}}\\ (1-b^2)^{-\frac{1}{2}}\end{vmatrix}.$$

Hence, as before, $\sin^{-1}x = x\,.\,\underset{0}{\overset{1}{\text{Av}}}\,(1-\theta^2x^2)^{-\frac{1}{2}}$.

We here need the error after the n term of a Binomial Expansion.

In the case of $(1+u)^p$ (see §§ VI, VIII, *Gazette*, Oct. 1903) it lies between 0 and

$$(p-1)_n\,.\,u^n\{(1+u)^n-1\},$$

$(p-1)_n$ denoting $\dfrac{(p-1)(p-2)\ldots(p-n)}{\lfloor n}$,

therefore the error in the case of $(1-\theta^2x^2)^{-\frac{1}{2}}$,

$$< (-\tfrac{3}{2})_n(-\theta^2x^2)^n\begin{vmatrix}(1-\theta^2x^2)^{-\frac{1}{2}}-1\\ 0\end{vmatrix};$$

$$\therefore \sin^{-1}x = \sum_0^n\frac{1\,.\,3\ldots\overline{2n-1}}{2\,.\,4\ldots 2n}\cdot\frac{x^{2n+1}}{2n+1}+\text{an error which lies between}$$

$$\frac{3\,.\,5\ldots\overline{2n+1}}{2\,.\,4\ldots 2n}\cdot x^{2n+1}\,.\,\text{Av}\,\theta^{2n}\{(1-\theta^2x^2)^{-\frac{1}{2}}-1\}\text{ and }0.$$

But $(1-\theta^2x^2)^{-\frac{1}{2}} < (1-x^2)^{-\frac{1}{2}}$;

$$\therefore \text{the error} < \frac{1\,.\,3\,.\,5\ldots 2n-1}{2\,.\,4\,.\,6\ldots 2n}\,x^{2n+1}\{(1-x^2)^{-\frac{1}{2}}-1\},$$

which diminishes indefinitely as n increases, if $x < 1$.

* The following inequality can be easily established: $\text{Av}\,.\,ab\,.\gtrless\text{Av}\,a\,.\,\text{Av}\,b$ according as the a's and b's increase together or contrariwise.

5. *The hyperbolic functions.*

Exactly similar work serves for these: for

$$\sinh x-\sinh y=2\sinh\frac{x-y}{2}.\cosh\frac{x+y}{2}$$

$$=2\tanh\frac{x-y}{2}.\cosh\frac{x-y}{2}\cosh\frac{x+y}{2},$$

and therefore $$<(x-y)\left|\begin{matrix}\cosh x\\ \cosh y\end{matrix}\right|.$$

$$\cosh x-\cosh y=2\sinh\frac{x-y}{2}\sinh\frac{x+y}{2}<(x-y)\left|\begin{matrix}\sinh x\\ \sinh y\end{matrix}\right|,$$

$$\tanh x-\tanh y=\frac{\sinh(x-y)}{\cosh x\cosh y}<(x-y)\left|\begin{matrix}\operatorname{sech}^2 y\\ \operatorname{sech}^2 x\end{matrix}\right|.$$

The method suggests the following answer to the difficult question 'what does "expansion" mean?'

The idea of continuity leads naturally to the expression of the increase of a function as a definite integral: expansion in a power series is the simplest form of integration by parts.

II.—EXPANSION OF FUNCTIONS IN GENERAL.

To prove that if f represents a function satisfying certain conditions for all values of the variable from a to $a+x$, then $f(a+x)$ can be expanded in powers of x.

Assume that, as in § I., $fu-fv<(u-v)\left|\begin{matrix}f'u\\ f'v\end{matrix}\right|$ where f' is another function which we call the *derived* function of f.

Now in the special cases investigated above $f'u$ was always greater or always less than $f'v$; in other words, f' was throughout an increasing or throughout a diminishing function. But this condition may not be fulfilled (*e.g.* $\cos x$ is diminishing from 0 to π, increasing from π to 2π, diminishing from 2π to 3π, etc.). In this general article slight modifications are introduced to provide for such cases.

If $x_1, x_2 \ldots$ are *any* intermediate values of the variable between x_0 and x_N, we obtain as before $fx_N-fx_0<\Sigma(x_i-x_{i-1})\left|\begin{matrix}f'x_i\\ f'x_{i-1}\end{matrix}\right|$; but the upper limit sometimes has $f'x_i$ and sometimes $f'x_{i-1}$.

Now choose the simplest series of values for x_i. Let x_0 be a, and let x_i be $a+\frac{ix}{N}$, so that the intermediate values divide the whole range a to $a+x$ into N equal parts, and $x_i-x_{i-1}=\frac{x}{N}$.

Then $f(a+x)-fa<\frac{x}{N}\Sigma\left|\begin{matrix}f'x_i\\f'x_{i-1}\end{matrix}\right|+\frac{x}{N}\Sigma'\left|\begin{matrix}f'x_{i-1}\\f'x_i\end{matrix}\right|$, Σ including all values of i for which f' is increasing, and Σ' all values for which it is diminishing.

Now assume that the whole range consists of parts such that in the first f' is increasing, in the second diminishing, and so on alternately.

Then in any one of these parts the difference between the upper and lower limits of $\Sigma\frac{x}{N}\left|\begin{matrix}f'x_i\\f'x_{i-1}\end{matrix}\right|$ is $\frac{x}{N}(f'x_m \sim f'x_n)$ where x_m, x_n are the ends of the part.

If therefore f' is finite throughout and if the number of parts is finite, this difference between the sums vanishes when N is indefinitely increased.

Thus $f(a+x)-fa=\frac{x}{N}\Sigma f'x_i$ or, when N is infinite, $x.\overset{1}{\underset{0}{\mathrm{Av}}}f'(a+\theta x)$;

$$\therefore\ f(a+x)=fa+x\,\mathrm{Av}\,f'(a+\theta x). \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

If f' satisfies the same conditions as have been laid down for f,

$$f'(a+\theta x)=f'a+\theta x\,.\overset{\theta'=1}{\underset{\theta'=0}{\mathrm{Av}}}f''(a+\theta'.\theta x).$$

Hence, as in § I.,

$$f(a+x)=fa+xf'(a)+\frac{x^2}{\underline{2}}f''(a)+\ldots+\frac{x^{i-1}}{\underline{i-1}}f^{i-1}(a)+\text{an error}$$

$$x^i\,.\overset{\theta_1=1}{\underset{\theta_1=0}{\mathrm{Av}}}\theta_1^{i-1}.\overset{1}{\underset{0}{\mathrm{Av}}}\theta_2^{i-2}.\ldots.\overset{1}{\underset{0}{\mathrm{Av}}}\theta'_{i-1}.\overset{\theta_i=1}{\underset{\theta_i=0}{\mathrm{Av}}}f^i(a+\theta_1\theta_2\ldots\theta_i x).$$

Now, $\theta_1\theta_2\ldots\theta_i$ being a fraction, $a+\theta_1\theta_2\ldots\theta_i x<\left|\begin{matrix}a+x\\a\end{matrix}\right|$;

$\therefore$ the f^i lies between the greatest and least values of f^i in the range;

$$\therefore\ \text{the error}<\frac{x^i}{\underline{i}}\left|\begin{matrix}A\\B\end{matrix}\right|.$$

If this error vanishes when i is increased indefinitely, $f(a+x)$ can be expanded in powers of x.

Conditions. It will be seen that the conditions to which f is subject are reducible to the following:

(1) $fu-fv$ must lie between $(u-v)f'u$ and $(u-v)f'v$, at least when the difference $(u-v)$ is infinitely small, and the same condition must hold for the derived functions $f', f'', \ldots$.

(2) $\frac{x}{N}(f'x_m \sim f'x_n)$ must vanish for infinite N when x_m, x_n are the ends of increasing or diminishing periods of f'. (This will be satisfied if f' is everywhere finite.)

(3). The number of these periods must be finite; and conditions (2) and (3) must hold for all the derived functions.

A function satisfying these conditions is called 'continuous.'

If f' is always increasing or always diminishing f might be called 'simply continuous.'

Corollary 1. If $f'x$ is given over a range $x=a$ to $x=b$, $fb-fa$ is determined by (1) without ambiguity, and therefore fx is determined if its value at any one point of the range is fixed arbitrarily.

Corollary 2. The geometrical interpretations of the conditions are simple and interesting; but the ideas have intentionally been kept analytical. It will be noticed that 'points of inflexion' do not of themselves break 'continuity' or interfere with the expansibility of a function.

Corollary 3. To prove that, with certain conditions,

$$f\phi(a+x)-f\phi a=x\,.\,\overset{1}{\underset{0}{\mathrm{Av}}}\,f'\phi(a+\theta x)\,.\,\phi'(a+\theta x).$$

This means that, in the general inequality at the beginning of the article, x_i is taken to be any function ϕ of $\left(a+\frac{ix}{N}\right)$.

We have $$x_i-x_{i-1}<\frac{x}{N}\begin{vmatrix}\phi'_i\\ \phi'_{i-1}\end{vmatrix}.$$

If $\phi'_i\gtrless\phi'_{i-1}$ when $f'\phi_i\gtrless f\phi'_{i-1}$, no difficulty arises, but, consider that for some successive values of i this is not true.

In such a period, if $\phi'_i<\phi'_{i-1}$ while $f'\phi_i>f'\phi_{i-1}$,

$$f\phi_i-f\phi_{i-1}<\frac{x}{N}\begin{vmatrix}\phi'_{i-1}\,.\,f'\phi_i\\ \phi'_i\,.\,f'\phi_{i-1}\end{vmatrix},$$

$$i.e.\ <\frac{x}{N}\begin{vmatrix}\phi'_i\,.\,f'\phi_i+(\phi'_{i-1}-\phi'_i)\,.\,f'\phi_i\\ \phi'_{i-1}\,.\,f'\phi_{i-1}-(\phi'_{i-1}-\phi'_i)\,.\,f'\phi_{i-1}\end{vmatrix};$$

$\therefore$ if A is the greatest value of $f'\phi_i$ in the period,

$$f\phi_m-f\phi_n<\frac{x}{N}\Sigma\begin{vmatrix}\phi'_i\,.\,f'\phi_i+\\ \phi'_{i-1}\,.\,f'\phi_{i-1}-\end{vmatrix}(\phi'_{i-1}-\phi'_i)A\Big|,$$

where m, n are the values of i at the ends of the period;

and $$\frac{x}{N}\Sigma(\phi'_{i-1}-\phi'_i)A\equiv\frac{Ax}{N}(\phi'_m-\phi'_n)$$

which vanishes if ϕ' is always finite.

Hence $$f\phi(a+x)-f\phi a=x\,.\,\overset{1}{\underset{0}{\mathrm{Av}}}\,f'\phi(a+\theta x)\,.\,\phi'(a+\theta x)$$

if $f'\phi$ and ϕ' are finite through the range. W. N. ROSEVEARE.

(To be continued.)

CONSTRUCTIONS WITH STRAIGHT-EDGE AND DIVIDERS.

Problem 1. From a given point A of a straight line AB to draw a line making with AB an angle equal to a given angle CAD which has its vertex at A.

From AB and AD mark off equal lengths AE, AF; and let EF cut AC in G.

From FE produced mark off $EH = GF$. Then AH is the required line, the angle BAH being equal to the angle CAD in magnitude and sign.

The measure of the complexity of this construction is 13.

Problem 2. To bisect a straight line AB.

Through A draw any straight line, and mark off on it equal distances AC, CD, DE.

Produce EB to F making $BF = EB$.

Let CF cut AB in G. G is the mid point of AB.

Complexity = 15.

A modification of this construction, having the same complexity, is obtained if F is taken in BE produced instead of in EB produced, and F is joined to D instead of to C.

Problem 3. To draw a parallel to a given line AB through a given point C.

Produce CA to D making $AD = CA$. Produce DB to E making $BE = DB$. CE is the required line.

Complexity = 15.

Problem 4. At a given point A to erect a perpendicular to a given line AB.

Draw any line CBD cutting AB in B and make BC and BD each $= BA$. Produce CA to E making $AE = AB$.

Let EB and AD intersect in F. Mark off from EB the length $EG = FB$. AG is the required perpendicular.

Complexity = 24.

Problem 5. From an external point C to draw a perpendicular to a given straight line AB.

Draw CD to meet AB in D. On either side of D mark off from the given line lengths DE, DF each $= DC$. From CE cut off $CG = CF$. Let GF cut CD in H. From FG mark off $FK = HG$. Let CK meet AB in L. CL is the required perpendicular.

Complexity = 22.

Problem 6. To trisect a given straight line AB.

Through A draw ACD making $AC = CD$.

Produce DB to E making $BE = BD$.

Let CE intersect AB in F. From FA mark off $FG = FB$.

Then F and G are the points of trisection.

Complexity = 17.

Problem 7. Given an angle BAC to make an equal angle at A on the opposite side of AB.

Through A draw any line AD and mark off $AE=AD$ on AB. Let ED meet AC in F. Produce DE to G making $EG=FD$. Join AG and from it cut off $AH=AE$. Let EH meet AD in K. Produce EH to L making $HL=KE$. Then AL makes with AB an angle equal to CAB.

Complexity = 28.

Problem 8. From a given point A in a given line AB to draw a line making with AB an angle equal to a given angle CDE.

Produce AD to N making $DN=AD$, and produce DA to F making $AF=DA$. Draw any line FGH through F cutting AB in G and make $GH=FG$. Mark off from DE, $DK=DH$. Let KH meet DC in L. From HK mark off $HM=LK$. Produce NM to P making $MP=NM$. Then AP is the required line.

Complexity = 32.

Note. The above solutions aim at being as simple as possible from the point of view of *construction*, taking the same *postulates* as those employed in Dr. Macaulay's article in last October's *Gazette*. They are therefore quite different from those which would be chosen if one wished to make the *proofs* as simple as possible. The complexity of each construction has been estimated according to the principle of Lemoine's *Géométrographie*, counting *one* every time the straight edge is adjusted to one point, or a point of the dividers adjusted to a point or a line, and every time a line is drawn.

I would like to suggest in this connection that the distinction between the two aims in elementary geometry, on the one hand the demonstration of propositions, and on the other hand the discovery of constructions for given problems, might with advantage be more emphasized in our "New Geometries"; and I would further suggest that the latter is a better exercise for the tyro in geometry. Lemoine's numerical test of simplicity makes a convenient standard for roughly comparing the merits of two constructions.

This article has been suggested by that of Dr. Macaulay. The problems chosen are among the most fundamental ones that can be based on the chosen postulates. Euclid's propositions 2 and 3 of Bk. I. show that every problem soluble by the postulates of this article, is also soluble by Euclid's postulates; but the converse is, I think, not true.

Perhaps some reader of the *Gazette* may be able to send a simplest possible construction for the problem: *To draw by the aid of straight-edge and dividers alone, a line bisecting at right angles the line joining two given points*, in other words *To draw the radical axis of two points.* I have hitherto found no con-

struction whose complexity is less than 38; while with *straight edge and compasses* a construction of complexity 7 exists, and with *Euclid's postulates* one of complexity 9.

[An old paper of mine on this subject, referred to by Dr. G. B. Halsted in his note in the March number of the *Gazette*, appeared not in the *Edin. Math. Soc.* Proceedings, but in those of the *Phil. Soc. of Glasgow*.] R. F. MUIRHEAD.

P.S.—Since writing the above, I have had an opportunity of glancing through Hilbert's *Grundlagen der Geometrie* (ed. 1899) for the first time, and I find that in that most interesting work the author gives some constructions based on the postulates used in the foregoing article. One of his constructions is virtually the same as the above solution of Problem 3, though by fixing points in the given line the complexity is slightly increased.

Professor Hilbert also gives a discussion of the question as to what kinds of problems of construction can be solved with these postulates, and arrives at a criterion, from which it would appear that Euclid I. 22 cannot be solved by straight-edge and dividers alone. I had previously become convinced of the impossibility just mentioned but without being able to give an *a priori* reason for it. The same impossibility would exist in the case of the simpler problem :—to draw a line of given length from a given point to a given straight line.

MATHEMATICAL NOTES.

170. [V. a.] *Negative quantities* (a very elementary note).

Certain difficulties, which history shews to be real and serious, in obtaining a correct idea of negative quantities are sometimes inadequately dealt with in books on elementary algebra.

The resulting confusion of thought is apt to set up mischief, as the doctors say, at a much later stage; as the following quotation from a recent school book on Physics sufficiently indicates: "If u is less than $\frac{1}{2}r$, then v is negative and equal to $-v$."

In arithmetic we perform certain operations on numbers. These numbers are, to begin with, positive integers. The operations of Addition and Multiplication always produce positive integers if we commence with positive integers. The operation of dividing one positive integer by another does not always furnish a positive integer. Instead, however, of saying that, for example, it is impossible to divide 7 by 3 we introduce the notion of a Fraction.

The operation of subtracting one positive integer from another does not always furnish a positive integer. Are we to say that $3-7$ or -7 by itself calls for the performance of an impossibility?

In generalised Arithmetic or Arithmetical Algebra our symbols a, b, c, etc., still denote essentially positive quantities, and the same difficulty of interpretation occurs.

We next perceive that an extension of meaning of the signs + and − will enable us to attach a meaning to -7. The signs + and − may be taken to indicate opposite qualities in the objects represented. Thus if we are

reckoning up a person's debts we may denote a sum of £5 due from him by $+5$; and then a sum of ten pounds due *to* him may be denoted by -10.

We can now give a meaning to the previously unintelligible expression $3-7$, and we perceive that whether a particular quality may more conveniently be denoted by the $+$ or by the $-$ sign depends on the point of view from which the problem is considered, and that the choice is a matter of convention.

It is evidently possible to employ a symbol to denote not only a quantity but also the sign of that quantity. Thus we might denote by the symbol x a quantity whose value proved to be -5. In this case x is a negative quantity.

The advantage of employing symbols which contain their own sign is brought out by contrasting the statements.

ARITHMETIC.

A owes me £a. B owes me £b. I owe C £c.

The symbols a, b, c, all denote positive quantities.

On these transactions I am in debt £$(c-a-b)$ or in credit by £$(a+b-c)$ according as c or $a+b$ is the greater.

ALGEBRA.

Let the balances due to me from A, B, C be α, β, γ respectively; β, for instance, being negative and equal to $-b$ if in reality I owe £b to B. Then on these transactions I am worth £$(\alpha+\beta+\gamma)$, and this expression adequately represents the following arithmetical results:

I am in credit on the transaction	I am in debt on the transaction
$a+b+c$	$a+b+c$
$a+b-c$	$a+b-c$
$a-b+c$	$a-b+c$
$-a+b+c$	$-a+b+c$
$a-b-c$	$a-b-c$
$-a-b+c$	$-a-b+c$
$-a+b-c$	$-a+b-c$

as the case may be.

The unsatisfactory character of the definition "a quantity with the $-$ sign prefixed is called a negative quantity" (Baker and Bourne, p. 7) is apparent, and may be illustrated by the problem, "In 1905 A is a years old, B is b years old. Find the year in which A might truly say to B, 'I am twice as many years old as you.'"

If x denotes the interval in years between the present year and the year in question, x is an algebraical symbol. Let x be positive if the year in question is in the future.

a and b denote essentially positive quantities. We obtain the equation

$$x=a-2b,$$

and we may take the numerical values

$$a=37,\ b=6;\quad a=37,\ b=35.$$

The weak point in the definition quoted above may be further exposed by applying it to $-\log 5$ or to $\cos 120°$.

There is an interesting article in the *English Cyclopedia* by De Morgan on Negative Quantities, which anyone who finds a difficulty in clearing up his pupils' ideas on this point may perhaps find it worth while to consult.

C. S. JACKSON.

171. [L². 8. b.] *On the normals from a point to a conicoid.*

Consider the rectangular hyperbolas in the principal planes of the conicoid $\frac{x^2}{a^2}+\frac{y^2}{b^2}+\frac{z^2}{c^2}=1$ which pass respectively through the feet of the four normals from $(0, \eta, \zeta)$ to the section by $x=0$; the feet of the four normals from $(\xi, 0, \zeta)$ to the section by $y=0$; and the feet of the four normals from $(\xi, \eta, 0)$ to the section by $z=0$. Then it may be proved that the feet of the six normals to the conicoid itself from ξ, η, ζ lie on the common curve of intersection of the three right cylinders whose cross sections are these three rectangular hyperbolas.

For the six feet are the intersections of the conicoid with

$$\frac{\xi-x}{x/a^2}=\frac{\eta-y}{y/b^2}=\frac{\zeta-z}{z/c^2},$$

i.e. with

$$\frac{\eta-y}{y/b^2}=\frac{\zeta-z}{z/c^2};\quad \frac{\zeta-z}{z/c^2}=\frac{\xi-x}{x/a^2};\quad \frac{\xi-x}{x/a^2}=\frac{\eta-y}{y/b^2}.$$

But these are the three cylinders mentioned above, and are seen to have a common curve of intersection. Thus the proposed theorem follows.

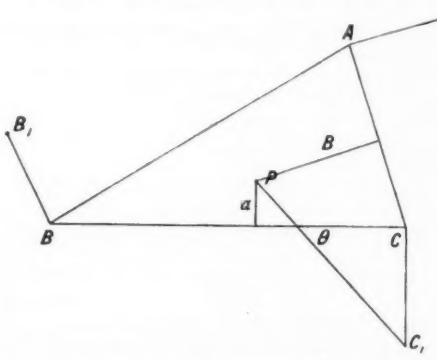
The centre of the conicoid and the point ξ, η, ζ itself are also on this curve.

H. L. TRACHTENBERG.

172. [K. 2. d.] *Three conics connected with a triangle.*

At C, A, B draw CC_1, AA_1, BB_1 respectively perpendicular to BC, CA, AB, such that $CC_1=AA_1=BB_1=v$.

Then as v varies, the locus of P, the centre of the circle $A_1B_1C_1$, is a cubic. For if P be α, β, γ, and PC_1 make θ with BC, and R be the radius PC_1,



$$\alpha+v=R\sin\theta \quad \text{and} \quad R\cos\theta=\frac{\beta+\alpha\cos C}{\sin C};$$

$$\therefore\ R^2=(\alpha+v)^2+\frac{(\beta+\alpha\cos C)^2}{\sin^2 C}.$$

$\therefore$ since two similar equations are true, we have, on eliminating R^2,

$$\alpha^2+2\alpha v+\frac{(\beta+\alpha\cos C)^2}{\sin^2 C}=\beta^2+2\beta v+\frac{(\gamma+\beta\cos A)^2}{\sin^2 A},$$

$$\alpha^2+2\alpha v+\frac{(\beta+\alpha\cos C)^2}{\sin^2 C}=\gamma^2+2\gamma v+\frac{(\alpha+\gamma\cos B)^2}{\sin^2 B}.$$

$\therefore$ eliminating v, we have the required locus,

$$\Sigma(\beta-\gamma)\left(\frac{a^2+\beta^2+2a\beta\cos C}{\sin^2 C}\right)=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

Again, the corresponding locus for perpendiculars drawn to CB, BA, AC at B, A, C respectively, is

$$\Sigma(\gamma-\beta)\left(\frac{a^2+\gamma^2+2a\gamma\cos B}{\sin^2 B}\right)=0. \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

These cubics intersect on the cubic

$$\Sigma(\beta-\gamma)\left(\frac{\beta^2+\gamma^2+2\beta\gamma\cos A}{\sin^2 A}\right)=0, \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

as is seen by subtracting one left-hand side from the other.

These cubics each consist of the line at infinity and a conic, the three conics being

$$\Sigma a\left\{\frac{a+\gamma\cos B}{\sin B}-\frac{\beta+\gamma\cos A}{\sin A}\right\}=0\,;\quad \Sigma a\left\{\frac{a+\beta\cos C}{\sin C}-\frac{\gamma+\beta\cos A}{\sin A}\right\}=0\,;$$

$$\Sigma a\left\{\frac{\beta+a\cos C}{\sin C}-\frac{\gamma+a\cos B}{\sin B}\right\}=0.$$

It will be noticed that the incentre and circumcentre lie on all three. Again, the bisector AI of the angle A, $\beta=\gamma$, meets the cubic (1) again in points on

$$\frac{\beta^2+\gamma^2+2\beta\gamma\cos A}{\sin^2 A}=\frac{\gamma^2+a^2+2\gamma a\cos B}{\sin^2 B},$$

i.e. on the straight line through the circumcentre, perpendicular to the side AB, and on the line at infinity.

Hence with similar results we have the fact that if AI meet the 3 lines through O perpendicular to BC, CA, AB respectively in

$$U_1,\ V_1,\ W_1,$$

and if the bisectors BI, CI meet them in

$$W_2,\ U_2,\ V_2$$

and

$$V_3,\ W_3,\ U_3,$$

then the points W_1, W_2, W_3 lie on the first conic,

the points V_1, V_2, V_3 „ the second conic,

and the points U_1, U_2, U_3 „ the third conic.

It will be noticed that P never goes to infinity in cases (1) and (2) in the conic parts of the loci. This can be seen geometrically (v being called negative when CC_1, BB_1, AA_1 are drawn in the opposite directions to those shewn in the figure). Thus the conics corresponding to (1) and (2) are closed.

It follows from the geometrical properties by which the cubics (1) and (2) were defined that there are 4 points connected with a triangle such that each is the centre of a pair of concentric circles, one of which cuts off equal portions CC_1, BB_1, AA_1 from the perpendiculars to CB, BA, AC at C, B, A respectively, and the other of which cuts off equal portions BB_1', CC_1', AA_1' from the perpendiculars to BC, CA, AB at B, C, A respectively; for the 4 points of intersection of the conics corresponding to (1) and (2) have this property.

H. L. TRACHTENBERG.

REVIEW.

Elements of Mechanics. Forty Lessons for Beginners in Engineering. By MANSFIELD MERRIMAN, Professor of Civil Engineering in Lehigh University. (New York, Wiley & Sons. London, Chapman & Hall. 1905. pp. 172. $1.)

Professor Merriman is the author of a number of text-books on several branches of applied mathematics, including a well-known treatise on the Method of Least Squares.

In the present volume his aim is to employ "the best methods of applied mechanics in the development of the fundamental principles and methods of rational mechanics." "To this end constant appeals are made to experience, by which alone the laws of mechanics can be established, ... and a system of units is employed with which every boy is acquainted." In the first 111 pages the leading principles of Statics are dealt with. The remainder of the book is devoted to Dynamics. Four hundred examples give ample occupation to the student.

The work is written in clear and concise terms, and the author appears thoroughly justified in expressing the hope that a student who has worked through the 400 problems will have laid a broad and strong foundation for his future progress. One point of detail offers an opening for discussion. Professor Merriman makes much use of the terms Inertia and Force of Inertia, following closely Newton's explanation of "Vis Inertiae." A teacher of elementary work has always to bear in mind that not everything which is lawful is expedient, and it may be permissible to remark that the expediency of using the terms Inertia, Force of Inertia, *in an elementary text-book*, is not established by the mere fact that Newton used them. Is not the statement "matter has Inertia" merely a poetical, or proverbial, or pithy mode of stating that Matter is distinguished from No-matter by the facts that Matter conforms to the "Laws of Motion"? Professor Merriman's own use of the terms Inertia and Force of Inertia is perfectly precise and unimpeachable; but does not the use made of so comfortable a word by some writers encourage a student to indulge in pseudo explanations such as that a body is hard to stop because it has so much inertia—a proposition which on investigation will be often found to belong to a transcendental branch of logic, known as feminine?

"Inertia" as an *explanation* of phenomena belongs to the class of philosophical terms which includes "vital force" as an *explanation* of the growth of living things. C. S. JACKSON.

ERRATA.

p. 181, line 10, *for* "portion" *read* "position."
p. 181, line 28, *for* "assumption" *read* "assumptions."
p. 181, line 2 up, *for* "diameter $x\pi$" *read* "diameter $\times \pi$."
p. 182, line 9, *for* "wave" *read* "move."
p. 182, line 23, *for* "vigorous" *read* "rigorous."

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A Key to Elementary Algebra. Part II. By W. M. BAKER and A. A. BOURNE. pp. 246. 1905. (Bell & Sons.)

Introduction to Analytic Geometry. By P. F. SMITH and A. S. GALE. pp. viii., 217. 5s. 6d. 1905. (Ginn & Co.)

Elements of the Kinematics of a Point and the Rational Mechanics of a Particle. By G. O. JAMES. pp. xii., 176. 8s. 6d. net. 1905. (Wiley; Chapman & Hall.)

Elementary Practical Mathematics. By H. A. STERN and W. H. TOPHAM. pp. viii., 110, viii. 1905. (Bell.)

Geometrical Conics. By G. W. CAUNT and C. M. JESSOP. pp. 80. 2s. 6d. 1905. (Ed. Arnold.)

Mathematical and Physical Papers. By SIR G. G. STOKES, Bart. Vol. 5. pp. xxv., 370. 15s. 1905. (Camb. Univ. Press.)

Euclid's Parallel Postulate: Its Nature, Validity, and Place in Geometrical Systems. By J. W. WITHERS. pp. vii., 192. 1905. (Open Court, Kegan Paul.)

Some Proofs of Newton's Theorem on Sums of Powers of Roots. A proof of Waring's Expression for $\Sigma\alpha^r$ in terms of the Coefficients of an Equation. By R. F. MUIRHEAD. (Proc. Edin. Math. Soc. Vol. xxiii. 1904-5.) pp. 9.

Practical Uses of the Product $\mathbf{ab}^2$ *in Vector Analysis.* By J. V. COLLINS. pp. ii. From *Am. Math. Monthly*, vol. xii., No. 2. 1905.

Correlation of Vector Analysis Notations. By J. V. COLLINS. pp. 4. From *Jahresbericht der Deutschen Mathematiker-Vereinigung*, Ap. 4, 1905.

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